**Modelling Juggling Patterns**

**Summary**

This project investigates the mathematics which underlies and describes the practise of juggling. I am going to look first at discrete models, and then at more general continuous models that can describe and manipulate patterns. Lastly I am going to suggest and briefly explore a new method for classifying patterns.

**Introduction**

Juggling has been around for many thousands of years, and so it is strange that any attempts to model it using mathematics have only happened very recently. Although it may seem like a trivial area to explore, the mathematics of juggling actually has many uses. Some of the applications are purely juggling related, for example inventing new tricks, whilst others branch out from juggling into other areas. This is because juggling is a quite simple and controlled process, making modelling it comparatively simple, but conversely it has interesting enough properties and complexities to make it helpful in looking at other areas, for example exploring the fundamentals behind bell ringing, knot theory, and the “many body problem”.

*http://www.juggling.org/papers/science-1/ and “The Mathematics of Juggling” by Burkard Polster*

**Notation**

The most commonly used notation to describe a juggling trick is called ‘siteswap’, and this is the notation that will be used for the next section in this project. However, this is also one of the simplest forms of notation, as it relies on a couple of broad assumptions about the juggling. A consequence of these properties is that juggling is being modelled as a discrete activity. The assumptions are:

J1) The throws are performed at discrete, evenly spaced moments in time, and each hand takes turns over these moments.

J2) Each pattern is periodic, and so can be repeated as many times as the juggler can manage.

J3) At most one ball is thrown and caught per beat, and it is the same ball that is both thrown and caught.

*Colin Wright, in person (lecture) and on his webpage*

Although these 3 properties constrict the patterns that can be performed, they also make the notation a lot simpler. This is good at first, as it allows us to take a fairly in depth look at some of the maths behind juggling without getting bogged down in complications such as hand positions etc.

A typical siteswap notation is a string of numbers, such as: 441. Each number is telling us how many beats to put the ball in the air for. As there is proportionality between time in the air and the height they are thrown, we will refer to it as height rather than time. We will assume for the sake of simplicity that there will be no throws higher than 9. Aside from the fact that this would be nearly impossible to perform, it also assures that 441 means 3 throws, of heights 4, 4 and 1, rather than 2 throws, of heights 44 and 1.

So, for example 441 is telling us that the first throw that occurs is of height 4, and then the second throw is of height 4, and the third throw is of height 1. More generally, a throw on beat i is of height ni.

Each throw is caught i+ni beats after it is thrown (this is quite an important result, and it is used later on). This means that if ni is even, then i+ ni will preserve the parity of i. As each hand takes turns (J1), then, for example, the right hand will be every even number, and the left hand every odd number. So an even n will be landing as the hand that threw it is about to have its turn, meaning that an even number throw is to the same hand, and odd throw is to the other hand.

All of this information can be represented in a diagram, 441:



This diagram is called a ladder diagram, and is read from top to bottom. Each coloured line represents one ball, and shows the path that the ball takes.



This is essentially a ladder diagram turned on its side, and is read from left to right.

Some examples

The most common juggling move is, contrary to common sense, not just a circle of balls, but is called the 3 ball cascade. It is written 3 in siteswap notation, and looks like:



The pattern 51 will have all the balls travelling in a circle around the jugglers head.



More generally for b balls, the trick can be written (2b-1)1.

**How is this juggled? Is it valid?**

So now we have a method of notation to write down any trick that is within the constraints of J1-3. But supposing my friend sends me one of his tricks, all I will see are a string of natural numbers n1n2… How do I know how many balls to use? And if I were to try and invent my own trick by just writing numbers down, how would I know that n1n2… is a valid sequence? For example if the sequence was not valid I would have to contradict any of J1, J2 or J3 in order to keep the balls in the air. Lastly, how do I know if my trick and my friend’s trick are actually the same?

Let us look at the last problem first, as this will assist us when defining what a sequence is. So, suppose we have two sequences x1x2… and y1y2…, and we want to know whether two people, X and Y, juggling sequence x and sequence y next to each other will look the same. Or, more specifically, the patterns can be the same, but asynchronous. This is what gives us a clue as to whether a sequence is the same or not. Let us look at the sequences A=441 and B=414. If we continue each sequence, as it would be when juggled, then A will extend to A’= 441441441 and B to B’= 414414414, then we can quickly see that sequence A occurs in A’ and B’, and sequence B also occurs in A’ and B’. This means that if we start juggling A, we will carry on to juggle the pattern A’, which will in turn mean that we have started juggling B, and, by J2, once we start juggling B, we will carry on forever on B. Hence we are now juggling A and B, so A=B.

A simpler way to look at this is to realise that A is a cyclic permutation (defined later) of B, so they are equal. More generally if a sequence X= x1x2…xi Belongs to the set of all cyclic permutation of Y where Y= y1y2…yj then X=Y then the two sequences are equal.

What if our first sequence, A, is extended to A’’= 4414? This is still the same sequence, but it is not a cyclic permutation of B. By looking at an extension of A’’, and B, we can see that they are the same. But is there a way to reduce A’’ into a most basic, or minimal, form? It turns out the easiest way is to just write out A’’ a couple of times, until we see the most simple pattern that is repeating. A condition when looking at cyclic permutations is that both patterns must be in minimal form.

Let us address the first problem, or how many balls a pattern requires. One method of finding the number of balls is to draw a ladder diagram, draw a horizontal line anywhere on the diagram, and then consider how many balls this horizontal line crosses. This will be the number of balls required. This method is fine for most tricks, as the number of balls will be small, and so the diagram is quick and simple to draw. However, for a more theoretical purpose, perhaps if we want to write a computer programme to simulate a given siteswap, a more theoretical approach is needed.

First we must take a new way of looking at a juggling sequence. If we take the beats, then a sequence could be looked at as a function . So your juggling pattern would look like …j(-1)j(0)j(1)j(2)… and, by J2, this would be infinite.

Let us also define:

And define: balls(j) = the number of balls needed to perform j

Theorem (the average theorem) *“The Mathematics of Juggling” by Burkard Polster pages 15, 16, 17*

B1) The number of balls needed is the average of the numbers in the sequence.

B2) Let j be a juggling function, if height(j) is finite then   
 = balls(j)

Where I is a sequence of i’s, = the cardinality of I, and the limit is over integer intervals, since .

Proof

B1 is an immediate consequence of B2, since the equation in B2 is essentially taking the average.

B2 is a bit more complex. Consider the set , and then let us consider a subset

Now for each throw, there is at least one that describes the height, j(i). Since height(j) is finite this means that balls(j) is finite. This fact will be shown after in a mini ‘proof’.

We know the sum of all is bounded from below by , since there are as many j(i) as i in I, and . And we can also see, by similar logic, that is bounded above by .This tells us that, dividing by and multiplying by balls(j):

Taking the limit of all parts of this:

So we get

Hence:

Proof that finite height(j) => finite balls(j).

By contradiction, suppose that we have balls(j)= and height(j)=k:. This will mean that we have an infinite number of balls in each hand.

We will perform the first throw, j(1), and then the second throw, and so on, until we have performed all throws .

Now we have two cases to consider.

Case 1:

There exists a throw such that. (Or that k<j(1)+1) This means that the ball thrown at i will land at a moment in time when we are trying to throw another ball, j(j(i)+i), hence contradicting J3.

Case 2:

Throw j(1) lands at j(1)+1, however we are throwing all j(i) up till j(j(1)+1)). As this will lead to us trying to throw and catch two separate balls we are contradicting J3.

Hence balls(j) must be finite.

So the only question left is: is new sequence valid? Well one way to check is to try juggling it! If we draw a ladder diagram, then we can see if any of J2, or J3 are contradicted. J2 will only be contradicted if the sequence is infinite, which it would still be able to juggle up to a point. J2 can also be contradicted if repeating the sequence directly after itself will lead to J3 being violated. Thus, the only thing we need really look for is if two balls try and land in one hand at the same time.

Showing this on a diagram, for example for a trick of the form n(n-1)…, such as 32… 

Here, we can see that two balls both land on the second L at the same time, thus violating J3.

However this is not the quickest method of determining whether or not the sequence is valid.

More mathematically, we can recall a throw of ni will land on beat i+ni , so if we define a new function (taking j to be the function from the previous section). So j\* is a function that maps every throw to the beat it will land on. If the set of j\*(i) is a permutation of the integers, then no j\*(n)=j\*(m), so no ball will land at the same time.

The permutation test *“The Mathematics of Juggling” by Burkard Polster page 22*

Let be a sequence such that and then s is a valid juggling sequence if and only if the function is a permutation of [p].

(in this definition = c : a = kb + c where )

Before proving this, we must define a cyclic shift, as it will be useful in the coming proof:

Let be a sequence with p and. Now let . The transformation from s to is called a cyclic shift of s.

Proof *“The Mathematics of Juggling” by Burkard Polster page 23*

with p and. In the case p = 1, there is nothing to show, so let p.

Let

First consider the effect of swapping beats i and i + d on s’. Only the ith and (i+d)th entry are changed.

The ith entry i + ai mod p, changes to i + (a(i+d) + d) mod p = i + d + a(i+d) mod p. The (i + d)th entry changes to (i + d) + (ai - d) mod p = i + ai mod p.

This means that the two entries just swap places and the transformed s’ contains all the elements of [p] if and only if the original s’ does.

Second, consider a cyclic shift on s’. This gives::=((p + ap-1, 1+ a0, 2+ a1, …,p - 2+ ap-2,) mod p). Clearly this new sequence has all elements of [p] if and only if s’ does.

By applying the flattening algorithm (given later) to a sequence of the form 1) m…m or 2) m(m-1)…, we can see that corresponds to a permutation, but does not.

Hence our result is proved.

The flattening algorithm *“The Mathematics of Juggling” by Burkard Polster page 20*

1. If s is a constant sequence, this is the sequence you want.
2. Use cyclic shifts until the maximum height, call it a, comes to rest at beat 0 and a non maximum height, call it b, comes to rest at beat 1. If then this is the sequence you want.
3. Perform the siteswap for beats 0 and 1. Let this sequence equal the new sequence, and return to step 1.

This algorithm transforms a juggling sequence into a constant sequence (i.e. 1, 2 or 3).

**How many patterns?**

The number of different site swaps that are of length n and using b balls is . Let no(b, n) be the number of patterns of length n and using b balls.[*http://www.its.caltech.edu/~juggling/mathematics.html*](http://www.its.caltech.edu/~juggling/mathematics.html)

This is surprisingly simple, however proving it rigorously is very hard, so we can use a double induction on this to make it plausible, rather than actually prove it.

Base case

For b=1, n=1, we are doing a siteswap that is of length 1, and using one ball. There is only one possible pattern, “1”. And.

Inductive step on b

Fix n=1, and fix some b. We want to show that the result is true for b+1, i.e. . This is true, as the only patterns that are of length one are constant sequences, i.e. 1 or 2.

Inductive step on n

Fix b, and fix some n. Now suppose . We want to show that . This is equivalent to which, by the inductive hypothesis, is .

This makes sense, since if you have every different site swap of length n for a number of balls, and then you want to add another digit to your sequence, for each sequence you will have a number of different choices for the next digit equal to the number of balls you have at your disposal.

**A different model**

Claude Shannon (1916-2001) was a great mathematician. He did lots of work in many areas, most famously for Boolean algebra and electronics, however it is a much less known fact that he was an avid juggler. He built the first known juggling machine, and was considered the first person to really apply maths to juggling.

For this section we are going to move beyond discrete juggling, and look at a more realistic model.

We will define juggling uniformly to have the following properties:

U1) The time any ball is held, or the dwell time, is a constant, d.

U2) The time any ball is in the air, or the flight time, is a constant, f.

U3) The time a hand is empty, or the vacant time, is a constant, v.

Note that U1 is relevant from both the point of view of the hand, and the point of view of the ball. Also that U3 is a consequence of U2, or U2 is a consequence of U3, depending on which point of view you start from.

For a very simple example, throwing one ball up and down in one hand (siteswap notation: 20). From the balls point of view, suppose it is held for one second, and then in the air for 2 seconds before it lands again. Therefore, clearly, the vacant time of the hand will have to be 2. The same applies if we start from the hands point of view.

As an aside, now that we are not following J1-3, it is interesting to note that a consequence of U1-3 is an underlying symmetry between hands and balls. The best way to see this is to imagine we have juggling balls that bounce on the floor, and we are juggling them by bouncing them on the floor from one hand to another. If we (hypothetically) have 3 hands, and 2 balls, and all that you can see are hands and balls, then in certain patterns it could almost appear that the balls are juggling the hands. This is shown by the following diagram:



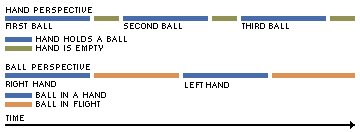
Shannon’s first Juggling theorem *http://www2.bc.edu/~lewbel/Shannon.html*

(f+d)h=(v+d)b

Where f is the time a ball spends in the air, d is the time a ball spends in a hand, v is the time a hand is vacant, n is the number of balls juggled, and h is the number of hands.

The left hand side of this formula is looking at the time a ball takes to return to where it started from the balls point of view. f+d is adding the total time the ball is in the two different states, and multiplying it by h repeats the total time for one beat by the number of hands. This is equated to the right hand side, which is from a hands point of view. v+d is adding the time the hand is in the two different states. This is then multiplied by b, so that the time it is in the two different states happens for each ball.

This is a diagram that helps explain what is happening:



*Re:f http://www2.bc.edu/~lewbel/Shannon.html*

Proof *“The Mathematics of Juggling” by Burkard Polster pages 98 and 99*

Let h be the number of hands.

First consider a ball that is part of a sequence. Suppose that the sequence has been performed a number of times such that the ball has been caught h times.

There are h hands, so the ball has been in a hand h+1 times (as it was held at the start as well as being caught). This means that it must have been in at least one of the hands (call it the left) more than once.

So in between two times that the left hand caught the ball, define the number of other times it was caught as m. By U1 and U2 we know that each throw has a time of d+f. The ball is caught m+1 times from left hand back to left hand. Hence (m+1)(f+d) time will pass. Over this time, the left hand will make n further catches. This will take (n+1)(v+d) time, using similar logic as before on U1 and U2. Since the time for the ball to get back to the left hand, and the tie for all the intervening throws of the left hand will be the same, we can conclude that (m+1)(f+d)= (n+1)(v+d) (\*).

Let g = hcf(m+1, n+1), p = and q = . This is basically simplifying the ratio of the two lengths of times into the simplest form.

This means we have (m+1)(f+d)= (n+1)(v+d) 🡪 p(f+d)= q(v+d). So we have derived the equation, and now we just need to show that p=h and q=b.

So consider a time interval of length p(f+d), and make the time interval such that no ball is caught at the beginning of the interval. Balls will be caught at times t1, t2, t3… . At time ti, si hands catch si balls, as two balls cannot be caught in one hand. Due to the length of our interval, each ball will be caught p times, and each hand will make q catches. From this we can conclude that: . Combining this with (\*) gives us our desired result.

What use is this theorem? Well it can tell you the frequency of a juggling pattern. The frequency of one of the hands throwing is v+d. b and h are known, and we can let b>h. Suppose we want to juggle this pattern with a fixed f, then we can manipulate this theorem to give us . So by varying v and d, we can see the maximum and minimum frequencies (when d=0 and when v=0 respectively).

**How to classify patterns/tricks**

Is there a way to classify tricks into different classes?

We could just order the tricks with respect to the number of balls they require. This certainly works, but is somewhat trivial, and does not lead us anywhere.

We could order them by some kind of numerical method, involving the siteswap numbers, but many tricks such as Mills Mess have no siteswap notation, as your hands move as well as the balls. So this method is no good, as it doesn’t take into account all tricks.

Take the simplest trick, not the easiest to perform, but the easiest to understand, is just the balls making a circle (for b balls, (2b-1)1). The shape they describe is a circle, with all the balls following after each other. So this trick is in the shape of a single circle.



Now take the normal three ball cascade (siteswap 3). Here the balls follow a sideways figure of eight shape, or a circle that has been twisted to create a figure of eight shape, and each ball follows the ball in front round. So the shape they make is also in the form of a single circle, albeit a transformed circle.



In basic four ball juggling (siteswap 4), one hand throws two balls one after the other in a circular pattern, and the other hand throws the other two balls in a circular pattern. Two balls follow each other, and the other two follow a different path. So this trick requires two circles to describe it.



Now let us look at Mills Mess. Here the balls move backwards and forwards. At no point do they come onto the same path, so this would require 3 circles to describe.



In this diagram, the hands should also be moving, however including movement lines for the hands would be unnecessary and confusing.

So you could describe each trick, rather than with siteswap, by how many circles it takes to define them, and how the circles are transformed. A trick of order 1 would require one circle, and order 2 would require two circles, etc.

Some examples:

|  |  |  |
| --- | --- | --- |
| Order 1 | Order 2 | Order 3 |
| 3 | 4 | “21” |
| 51 | 46 | “2” |
| 441 | 24 | 4413 |

(these are written only in siteswap, and so not every trick is fully defined, those that would need more information are in quotation marks)

*Reference: idea taken and expanded from Colin Wright, in person (lecture) and on his webpage*

**Conclusion**

This project has analysed simple juggling patterns with discrete and continuous methods. Each method has yielded varying results. The discrete method enabled me to generate new simple tricks, but stayed rigidly within a framework. The continuous method seemed that it would allow more scope for development if the restrictions were removed one by one. Finally I briefly looked at different methods of classifying, and decided on one particular method to pursue in a bit more depth.

In my second year at college I attended a lecture about the mathematics of juggling run by Collin Wright. This was central to my project, and I have taken the lecture material and tried to expand upon it where appropriate. I also recently received a book, “The Mathematics of Juggling” by Burkard Polster, and this has allowed me to see rigorous theorems and proofs, which I have expanded on and rewritten to improve the clarity where I can. I looked at other websites, mentioned in the bibliography, in order to give a more comprehensive look at the mathematics that underlies juggling.

**Bibliography**

http://www.juggling.org/papers/science-1/

Colin Wright, in person (lecture) and on his webpage, http://www.solipsys.co.uk/new/JugglingTalkSummary.html?JugglingTalk

“The Mathematics of Juggling” by Burkard Polster

http://www2.bc.edu/~lewbel/Shannon.html

<http://www.its.caltech.edu/~juggling/mathematics.html>